Joint Coordinate Method for Analysis and Design of Multibody Systems : Part 1. System Equations

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This paper presents a method, known as the joint coordinate method, for the computer aided design process of multibody systems. The relationship between joint and absolute velocities is described by a linear velocity transformation matrix representing the system kinematics. Joint coordinate method can generate a minimal set of equations of motion in terms of the generalized joint accelerations. This method can also yield reaction and actuator forces acting at the kinematic joints that are necessary for forward and inverse dynamics analyses. Applications of this method to static equilibrium and design sensitivity analyses are also studied. Comparisons between the absolute and joint coordinate formulations are given in terms of their computational efficiency.

Key Words: Joint Coordinates, Velocity Transformation, Equations of Motion, Reaction Force, Actuator Force, Static Equilibrium, Design Sensitivity

1. Introduction

In recent years, the derivation of the equations of motion for multibody systems has been the topic of many research activities. Many researchers have preferred to derive these equations in terms of a large number of absolute coordinates due to their simplicity and ease of manipulation (Orlandea, Chace and Calahan, 1977; Wehage, 1982; Nikravesh and Chung, 1982; Nikravesh, 1988; Haug, 1989). The drawback of using absolute coordinates is that the equations of motion form a set of mixed algebraic-differential equations, which results in a computationally inefficient algorithm.

Another approach is to derive the equations of motion in terms of a minimal set of generalized accelerations. One popular method in this category is the so-called joint coordinate method based on a linear velocity transformation process (Sheth and Uicker, 1972; Jerkovsky, 1978; Wittenburg, 1988; Kim and Vanderploeg, 1986; Wehage, 1988; Nikravesh and Gim, 1993). Some comparisons on this group of studies have been performed (Nikravesh, Gim, Arabyan and Rein, 1989; Rein, 1989). In this method, absolute coordinates are used to define the position of each body, the kinematic joints between the bodies, and the forces acting on the bodies. Next, a set of relative joint coordinates is defined for the system, and then the equations of motion are converted from absolute accelerations to relative joint accelerations, which yields a minimal set of differential equations equal in number to the number of degrees of freedom of the system.

For a computer aided design process of multibody systems, the design sensitivity analysis has been studied in past years (Haug, Wehage and

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Mani, 1984; Hsieh and Arora, 1984; Krisnawami and Bhatti, 1984; Chang and Nikravesh, 1985; Paeng and Arora, 1989; Nalecz, 1989). Since most of the studies were based on the absolute coordinates, they did not provide a computationally efficient algorithm for any real design applications. For the design sensitivity analysis, the computational efficiency is very important because both the equations of motion and the equations of design sensitivity are solved repeatedly in the optimization iteration process. Since the joint coordinate method produces a minimal set of equations of design sensitivity, its usage can provide a more computationally efficient algorithm.

In this study, the relationship between the joint and absolute velocities is first described by a linear velocity transformation matrix, which is constructed from block matrices representing the system kinematics. Joint coordinate method then generates a minimal set of equations of motion in terms of the generalized joint accelerations (Nikravesh and Gim, 1993). The derivation of reaction and actuator forces at a kinematic joint is studied. The equations of static equilibrium are derived in terms of the joint coordinates. An application of the joint coordinate method to the design sensitivity analysis is also studied. Finally the comparison of computational efficiency between the absolute and joint coordinate formulations is given.

2. Kinematics

A multibody system is defined as an assembly of rigid or deformable bodies interconnected by a set of kinematic joints under the action of external and internal forces and/or moments. A schematic representation of a multibody system is shown in Fig. 1.1. In order to specify the position of a rigid body in a global non-moving xyz coordinate system, it is sufficient to specify the spatial location of the origin (center of mass) and the angular orientation of a body-fixed $\xi\eta\zeta$ coordinate system as shown in Fig. 1.2. For the ith body in a multibody system, vector $\boldsymbol{q}_i$ denotes a vector of

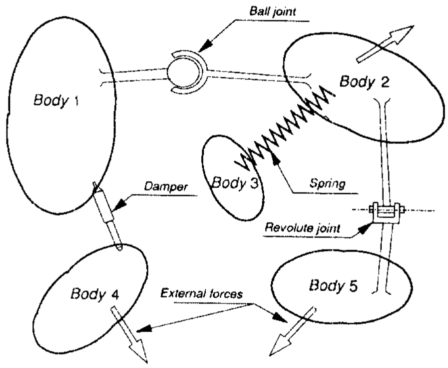


Fig. 1.1 A systematic representation of the multibody system

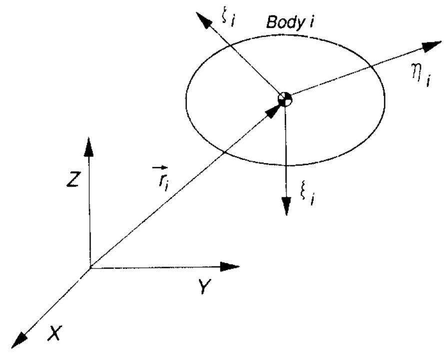


Fig. 1.2 Body-fixed and global coordinate system

coordinates which contains a vector of Cartesian translational coordinates $\boldsymbol{r}_i$ and a vector of rotational coordinates. A vector of veloctities for body i is defined as $\boldsymbol{v}_i$, which contains a vector of translational velocities $\dot{\boldsymbol{r}}_i$ and a vector of angular velocities $\boldsymbol{\omega}_i$. A vector of accelerations for this body is denoted by $\dot{\boldsymbol{v}}_i$ which contains $\ddot{\boldsymbol{r}}_i$ and $\dot{\boldsymbol{\omega}}_i$. For a multibody system containing b bodies, vectors of coordinates, velocities, and accelerations are defined as $\boldsymbol{q}$, $\boldsymbol{v}$, and $\dot{\boldsymbol{v}}$ which contain the elements of $\boldsymbol{q}_i$, $\boldsymbol{v}_i$, and $\dot{\boldsymbol{v}}_i$, respectively, for $i = 1, \cdots, b$.

For a multibody system with open kinematic loops, as the example shown in Fig. 1.3, the relative configuration of two adjacent bodies is defined by one or more relative coordinates, kn-

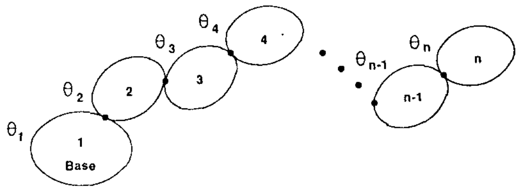


Fig. 1.3 An open-loop system

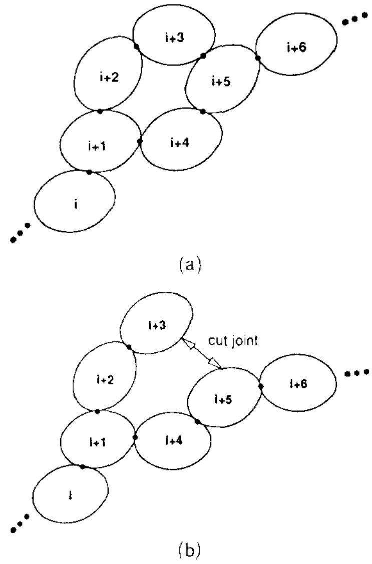


Fig. 1.4 A system containing a closed-loop; (a) a closed-loop and (b) its reduced open-loop representation

own as joint coordinates. These coordinates equal in number to the number of relative degrees of freedom between the bodies. If a vector of joint coordinates for the system is defined as $\boldsymbol{\theta}$, then it contains all of the joint coordinates and the absolute coordinates of a base (reference) body if the base body is not the ground (floating base body). This vector has a dimension equal in number to the number of degrees of freedom of the system. A vector of joint velocities is defined as $\dot{\boldsymbol{\theta}}$, the first time derivative of $\boldsymbol{\theta}$. It can be shown that there is a linear transformation between $\dot{\boldsymbol{\theta}}$ and $\boldsymbol{v}$ as (Jerkovsky, 1978; Wittenburg, 1988; Kim and Vanderploeg, 1986)

$$\boldsymbol{v} = \boldsymbol{B}\dot{\boldsymbol{\theta}}, \tag{1.1}$$

where matrix $\boldsymbol{B}$ is denoted as a veloctiy transformation matrix between the joint and absolute velocities. This matrix can be determined based on the kinematics and the topology of a multibody system.

For a multibody system containing closed kinematic loops, as the example shown in Fig. 1.4(a), each closed-loop is cut at one of the kinematic joints in order to obtain a reduced open-loop system as shown in Fig. 1.4(b). For this reduced system, joint coordinates are defined as for any open-loop system. It is clear that if we now close this system at the cut joint(s), the joint coordinates will no longer be independent; i.e., for each closed-loop there exist one or more holonomic constraints. If a vector of independent joint coordinates with a dimension equal in number to the number of degrees of freedom of the system is denoted as $\boldsymbol{\theta}_{(i)}$, chosen as a subset of vector $\boldsymbol{\theta}$, it requires no kinematic constraints to describe the system. If the first time derivative of $\boldsymbol{\theta}_{(i)}$ is denoted as a vector of joint velocities $\dot{\boldsymbol{\theta}}_{(i)}$, then a linear velocity transformation can be defined as (Nikravesh and Gim, 1993)

$$\dot{\boldsymbol{\theta}} = \boldsymbol{E}\dot{\boldsymbol{\theta}}_{(i)}, \tag{1.2}$$

where matrix $\boldsymbol{E}$ is denoted as a velocity transformation matrix between $\dot{\boldsymbol{\theta}}_{(i)}$ and $\dot{\boldsymbol{\theta}}$.

Kinematic joints interconnecting bodies in a multibody system are described by a set of m independent holonomic constraints in terms of the absolute coordinates as (Nikravesh, 1988)

$$\boldsymbol{\Phi}(\boldsymbol{q}) = \boldsymbol{0}. \tag{1.3}$$

The first and second time derivatives of Eq. (1.3) yield

$$\dot{\boldsymbol{\Phi}} \equiv \boldsymbol{D}\boldsymbol{v} = \boldsymbol{0}, \tag{1.4}$$

$$\ddot{\boldsymbol{\Phi}} \equiv \boldsymbol{D}\dot{\boldsymbol{v}} - \boldsymbol{\gamma} = \boldsymbol{0}, \tag{1.5}$$

where $\boldsymbol{D}$ is the Jacobian matrix of the constraints in Eq. (1.3) and $\boldsymbol{\gamma} = -\dot{\boldsymbol{D}}\boldsymbol{v}$.

In an open-loop system, if the joint coordinate method is employed, the constraints of Eq. (1.3) are no longer needed, since the joint coordinates themselves represent the relative configuration of kinematic joints in the system. For a closed-loop system, however, some independent holonomic constraints are necessary and can be written as (Nikravesh and Gim, 1993)

$$\boldsymbol{\Psi}(\boldsymbol{\theta})=\mathbf{0}, \tag{1.6}$$

The first and second time derivatives of Eq. (1.6) are written as

$$\dot{\boldsymbol{\Psi}}\equiv \boldsymbol{C}\dot{\boldsymbol{\theta}}=\mathbf{0}, \tag{1.7}$$

$$\ddot{\boldsymbol{\Psi}}\equiv \boldsymbol{C}\ddot{\boldsymbol{\theta}}-\boldsymbol{\varkappa}=\mathbf{0}, \tag{1.8}$$

where $\boldsymbol{C}$ is the Jacobian matrix of the constraints in Eq. (1.6) and $\boldsymbol{\varkappa}=-\dot{\boldsymbol{C}}\dot{\boldsymbol{\theta}}$.

3. Equations of Motion

In this study, the equations of motion are first expressed in terms of the absolute accelerations, which results in a large set of mixed differential -algebraic equations. The equations of motion are then converted from absolute accelerations to relative joint accelerations. For open-loop systems, this process is done in one step and the resultant equations are equal in number to the number of degrees of freedom of the system. The conversion process can be performed in two steps for systems containing closed-loops. The first step of the conversion results in a set of mixed algebraic-differential equations, since the joint coordinates are not independent of each other. The second conversion step can be applied to produce a minimal set of differential equations of motion in terms of an independent set of generalized joint accelerations.

When the absolute coordinate system is used, the equations of motion are written as (Nikravesh, 1988)

$$\boldsymbol{M}\dot{\boldsymbol{v}}-\boldsymbol{D}^T\boldsymbol{\lambda}=\boldsymbol{g}, \tag{1.9}$$

where $\boldsymbol{M}$ is the inertia matrix containing the mass and inertia tensor of all bodies, $\boldsymbol{\lambda}$ is a vector of m Lagrange multipliers, and $\boldsymbol{g}=\boldsymbol{g}(\boldsymbol{q}, \boldsymbol{v})$ contains the gyroscopic terms, and the forces and moments acting on the system. Note that Eqs. (1.9) and (1.5) represent a large set of differential-algebraic equations of motion for a multibody system. These equations have the same form whether the system contains open- or closed-loops.

For a multibody system with open kinematic loops (refer to Fig. 1.3), the above equations of motion are converted to a minimal set of differential equations equal in number to the number of degrees of freedom. By substituting Eq. (1.1) into Eq. (1.4) and knowing that $\dot{\boldsymbol{\theta}}$ is a vector of independent velocities, we get

$$\boldsymbol{DB}=\mathbf{0}. \tag{1.10}$$

The first time derivative of Eq. (1.1) gives the acceleration transformation formula as

$$\dot{\boldsymbol{v}}=\boldsymbol{B}\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}}. \tag{1.11}$$

Substituting Eq. (1.11) into Eq. (1.9), premultiplying by $\boldsymbol{B}^T$, and using Eq. (1.10) yield

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{\theta}}=\boldsymbol{f}, \tag{1.12}$$

where $\bar{\boldsymbol{M}}=\boldsymbol{B}^T\boldsymbol{M}\boldsymbol{B}$ and $\boldsymbol{f}=\boldsymbol{B}^T(\boldsymbol{g}-\boldsymbol{M}\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}})$. Equation (1.12) represents the equations of motion for an open-loop system when the number of selected coordinates is euqal to the number of degrees of freedom of the system.

For a multibody system containing closed kinematic loops (refer to Fig. 1.4(a)), the equations of motion in terms of the joint accelerations can be obtained either as a small set of algebraic-differential equations or as a minimal set of differential equations. For a closed-loop system, Eq. (1.12) is modified as (Nikravesh and Gim, 1993)

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{\theta}}-\boldsymbol{C}^T\boldsymbol{\nu}=\boldsymbol{f}, \tag{1.13}$$

where $\boldsymbol{\nu}$ is a vector of Lagrange multipliers associated with the constraints of Eq. (1.6). Equations (1.13) and (1.8) represent the equations of motion for a multibody system when the number of selected joint coordinates is greater than the number of degrees of freedom of the system.

The vector of Lagrange multipliers $\boldsymbol{\nu}$ in Eq. (1.13) can be eliminated in order to obtain a minimal set of equations of motion in terms of a set of independent joint accelerations. Since $\dot{\boldsymbol{\theta}}_{(i)}$ is a vector of independent velocities, substitution of Eq. (1.2) into Eq. (1.7) yields

$$\boldsymbol{CE}=\mathbf{0}. \tag{1.14}$$

The first time derivative of Eq. (1.2) gives

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{E}\ddot{\boldsymbol{\theta}}_{(i)} + \dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)}. \qquad (1.15)$$

Substituting Eq. (1.15) into Eq. (1.13), premultiplying by $\boldsymbol{E}^T$, and using Eq. (1.14) yield

$$\bar{\boldsymbol{M}}'\ddot{\boldsymbol{\theta}}_{(i)} = \boldsymbol{f}', \qquad (1.16)$$

where $\bar{\boldsymbol{M}}' = \boldsymbol{E}^T\bar{\boldsymbol{M}}\boldsymbol{E}$ and $\boldsymbol{f}' = \boldsymbol{E}^T(\boldsymbol{f} - \bar{\boldsymbol{M}}\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)})$. Equation (1.16) represents a minimal set of equations of motion describing the dynamics of a multibody system containing closed kinematic loops.

Jacobian matrix $\boldsymbol{C}$ from Eq. (1.13) and vector $\boldsymbol{\varkappa}$ from Eq. (1.8) can be found by introducing the constraints of the cut joints expressed as functions of $\boldsymbol{q}$; i.e.,

$$\boldsymbol{\Phi}^*(\boldsymbol{q}) = \boldsymbol{0}. \qquad (1.17)$$

The first and second time derivatives of the constraints yield

$$\dot{\boldsymbol{\Phi}}^* \equiv \boldsymbol{D}^*\boldsymbol{v} = \boldsymbol{0}, \qquad (1.18)$$

$$\ddot{\boldsymbol{\Phi}}^* \equiv \boldsymbol{D}^*\dot{\boldsymbol{v}} - \boldsymbol{\gamma}^* = \boldsymbol{0}, \qquad (1.19)$$

where $\boldsymbol{D}^*$ is the Jacobian matrix and $\boldsymbol{\gamma}^* = -\dot{\boldsymbol{D}}^*\boldsymbol{v}$. Substitution of Eq. (1.1) into Eq. (1.18) yields

$$\dot{\boldsymbol{\Phi}}^* \equiv \boldsymbol{D}^*\boldsymbol{B}\dot{\boldsymbol{\theta}} = \boldsymbol{0}.$$

If we eliminate the redundant rows of matrix $\boldsymbol{D}^*\boldsymbol{B}$ and compare the above equation with Eq. (1.7), the resultant matrix yields matrix $\boldsymbol{C}$ as

$$\boldsymbol{D}^*\boldsymbol{B} \rightarrow \boldsymbol{C}.$$

We can substitute Eq. (1.11) into Eq. (1.19) to get

$$\ddot{\boldsymbol{\Phi}}^* \equiv \boldsymbol{D}^*\boldsymbol{B}\ddot{\boldsymbol{\theta}} + \boldsymbol{D}^*\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}} - \boldsymbol{\gamma}^* = \boldsymbol{0}.$$

If we eliminate the redundant constraints and compare the above equation with Eq. (1.8), then we get vector $\boldsymbol{\varkappa}$ as

$$\boldsymbol{\gamma}^* - \boldsymbol{D}^*\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}} \rightarrow \boldsymbol{\varkappa}.$$

In order to find matrix $\boldsymbol{E}$ and vector $\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)}$ from Eq. (1.16), we can partition vector $\dot{\boldsymbol{\theta}}$ into dependent and independent sets, $\dot{\boldsymbol{\theta}}_{(d)}$ and $\dot{\boldsymbol{\theta}}_{(i)}$, and correspondingly we can partition the Jacobian matrix $\boldsymbol{C}$ into two submatrices $\boldsymbol{C}_{(d)}$ and $\boldsymbol{C}_{(i)}$. Then, Eqs. (1.7) and (1.8) can be written as

$$\boldsymbol{C}_{(d)}\dot{\boldsymbol{\theta}}_{(d)} + \boldsymbol{C}_{(i)}\dot{\boldsymbol{\theta}}_{(i)} = \boldsymbol{0}, \qquad (1.20)$$

$$\boldsymbol{C}_{(d)}\ddot{\boldsymbol{\theta}}_{(d)} + \boldsymbol{C}_{(i)}\ddot{\boldsymbol{\theta}}_{(i)} = \boldsymbol{\varkappa}. \qquad (1.21)$$

Equation (1.20) is written for $\dot{\boldsymbol{\theta}} = [\dot{\boldsymbol{\theta}}_{(i)}^T, \ \dot{\boldsymbol{\theta}}_{(d)}^T]^T$ as

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{C}_{(d)}^{-1}\boldsymbol{C}_{(i)} \end{bmatrix}\dot{\boldsymbol{\theta}}_{(i)}, \qquad (1.22)$$

where a proper selection of independent joint velocities guarantees that $\boldsymbol{C}_{(d)}$ is a nonsingular matrix (Nikravesh, 1988). Comparison of Eqs. (1.2) and (1.22) yields

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{C}_{(d)}^{-1}\boldsymbol{C}_{(i)} \end{bmatrix}. \qquad (1.23)$$

Equation (1.21) is rearranged for $\ddot{\boldsymbol{\theta}} = [\ddot{\boldsymbol{\theta}}_{(i)}^T, \ \ddot{\boldsymbol{\theta}}_{(d)}^T]^T$ as

$$\ddot{\boldsymbol{\theta}} = \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{C}_{(d)}^{-1}\boldsymbol{C}_{(i)} \end{bmatrix}\ddot{\boldsymbol{\theta}}_{(i)} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}_{(d)}^{-1}\boldsymbol{\varkappa} \end{bmatrix}. \qquad (1.24)$$

By comparing Eqs. (1.15) and (1.24), and using Eq. (1.23), we find

$$\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}_{(d)}^{-1}\boldsymbol{\varkappa} \end{bmatrix}.$$

It has been shown that for a multibody system containing closed-loops, Eqs. (1.13) and (1.8), or Eq. (1.16) provide a computationally efficient algorithm, while Eq. (1.12) is an efficient formulation for an open-loop system. A process for a systematic generation of vectors $\boldsymbol{g}$ and $\boldsymbol{\gamma}$, and matrices $\boldsymbol{M}$ and $\boldsymbol{D}$ is described in (Nikravesh, 1988). Since matrix $\boldsymbol{B}$ is constructed in explicit form in terms of a vector of absolute coordinates $\boldsymbol{q}$, matrix $\dot{\boldsymbol{B}}$ is expressed explicit form as a function of $\boldsymbol{q}$ and $\boldsymbol{v}$ (Kim and Vanderploeg, 1986).

4. Reaction Forces

In some applications, it may be required to determine reaction forces acting at a kinematic joint. Since a vector of Lagrange Multipliers $\boldsymbol{\lambda}$ is no longer available from Eqs. (1.12), (1.13), and (1.16), conventional way of finding the reaction forces using $\boldsymbol{\lambda}$ cannot be used here. If we use the system topology and kinematic properties, however, the reaction forces can be determined. In order to find reaction forces and moments, $\boldsymbol{f}_j^{(c)}$ and $\boldsymbol{n}_j^{(c)}$ respectively, at joint j, we consider a subsystem which starts from body j toward a leaf body (not toward the base body) as shown in Fig. 1.5. By applying vectors $\boldsymbol{f}_j^{(c)}$ and $\boldsymbol{n}_j^{(c)}$ at the attach-

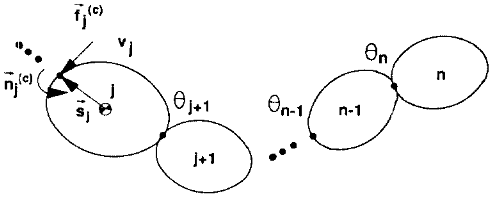


Fig. 1.5 A subsystem for reaction forces acting at joint j

ment point of joint j on body j, joint j is eliminated, but instead body j is considered as a floating base body.

If vectors of absolute and joint velocities of the subsystem are denoted as $\boldsymbol{v}_s$ and $\dot{\boldsymbol{\theta}}_s$ respectively, then a linear transformation between $\boldsymbol{v}_s$ and $\dot{\boldsymbol{\theta}}_s$ is written as

$$\boldsymbol{v}_s=\boldsymbol{B}_s\dot{\boldsymbol{\theta}}_s, \tag{1.25}$$

where subscript s stands for the subsystem. If a vector of absolute velocities of body j is defined as $\boldsymbol{v}_j$ and vectors of absolute and joint velocities of the substytem, without body j, are defined as $\boldsymbol{v}_s'$ and $\dot{\boldsymbol{\theta}}_s'$ respectively, we have $\boldsymbol{v}_s=[\boldsymbol{v}_j^T,\ \boldsymbol{v}_s'^T]^T$ and $\dot{\boldsymbol{\theta}}_s=[\boldsymbol{v}_j^T,\ \dot{\boldsymbol{\theta}}_s'^T]^T$. Then Eq. (1.25) can be written as

$$\begin{bmatrix}\boldsymbol{v}_j\\ \boldsymbol{v}_s'\end{bmatrix}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{B}_{si}' & \boldsymbol{B}_{s2}'\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_j\\ \dot{\boldsymbol{\theta}}_s'\end{bmatrix} \tag{1.26}$$

The equations of motion for the subsystem are written as

$$\boldsymbol{M}_s\dot{\boldsymbol{v}}_s-\boldsymbol{D}_s^T\boldsymbol{\lambda}_s=\boldsymbol{g}_s+\begin{bmatrix}\boldsymbol{g}_j^{(c)}\\ \boldsymbol{0}\end{bmatrix}, \tag{1.27}$$

where $\boldsymbol{g}_j^{(c)}$ is a vector of reaction forces acting at the center of mass of body j. This vector is expressed as a function of the reaction forces and moments acting at joint j; i.e.,

$$\boldsymbol{g}_j^{(c)}=\begin{bmatrix}\boldsymbol{f}_j^{(c)}\\ \boldsymbol{n}_j^{(c)}+\tilde{\boldsymbol{s}}_j\boldsymbol{f}_j^{(c)}\end{bmatrix}, \tag{1.28}$$

where vector $\tilde{\boldsymbol{s}}_j$ is a position vector from the center of mass of body j to the attachment point of joint j. Now we consider the subsystem containing open-loops or reduced open-loops with cut joints (refer to Figs. 1.3 or 1.4(b) respectively). Premultiplying Eq. (1.27) by $\boldsymbol{B}_s^T$ and realizing $\boldsymbol{B}_s^T\boldsymbol{D}_s^T=\boldsymbol{0}$ yield

$$\boldsymbol{B}_s^T[\boldsymbol{g}_j^{(c)T},\ \boldsymbol{0}^T]^T=\boldsymbol{B}_s^T(\boldsymbol{M}_s\dot{\boldsymbol{v}}_s-\boldsymbol{g}_s).$$

By substituting $\boldsymbol{B}_s$ described in Eq. (1.26) into the above equation, we have

$$\begin{bmatrix}\boldsymbol{I} & \boldsymbol{B}_{s1}'^T\\ \boldsymbol{0}^T & \boldsymbol{B}_{s2}'^T\end{bmatrix}\begin{bmatrix}\boldsymbol{g}_j^{(c)}\\ \boldsymbol{0}\end{bmatrix}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{B}_{s1}'^T\\ \boldsymbol{0}^T & \boldsymbol{B}_{s2}'^T\end{bmatrix}(\boldsymbol{M}_s\dot{\boldsymbol{v}}_s-\boldsymbol{g}_s).$$

The first row of the above equation therefore becomes

$$\boldsymbol{g}_i^{(c)}=[\boldsymbol{I},\ \boldsymbol{B}_{s1}'^T]\ (\boldsymbol{M}_s\dot{\boldsymbol{v}}_s-\boldsymbol{g}_s) \tag{1.29}$$

For an open- or closed-loop system, reaction forces and moments, $\boldsymbol{f}_j^{(c)}$ and $\boldsymbol{n}_j^{(c)}$, acting at joint j can be determined from Eqs. (1.28) and (1.29). For a closed-loop system, vector $\boldsymbol{g}_s$ should include the reaction forces and moments of a cut joint(s) acting at its adjacent bodies. The reaction forces and moments of a cut joint(s) can be determined by $\boldsymbol{C}^T\boldsymbol{\nu}$ from Eq. (1.13). Matrix $[\boldsymbol{I},\ \boldsymbol{B}_{s1}'^T]^T$ is the submatrix corresponding to body j as a floating base body.

5. Actuator Forces

In order to manipulate robotic devices or to control other types of mechanical systems, the force (or the moment) of the actuator should be determined based on the known configuration and motion of the system. Joint coordinate method provides a powerful tool for this process. Vector $\boldsymbol{g}$ is divided into two sets of vectors $\boldsymbol{g}^{(a)}$ and $\boldsymbol{g}^{(na)}$ corresponding to the actuator forces and non-actuator forces.

For an open-loop system, a vector of actuator forces can be obtained from Eq. (1.12) as

$$\boldsymbol{\tau}=\bar{\boldsymbol{M}}\ddot{\boldsymbol{\theta}}-\boldsymbol{f}^{(na)}, \tag{1.30}$$

where $\boldsymbol{\tau}=\boldsymbol{B}^T\boldsymbol{g}^{(a)}$ and $\boldsymbol{f}^{(na)}=\boldsymbol{B}^T(\boldsymbol{g}^{(na)}-\boldsymbol{M}\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}})$ are vectors of actuator forces and non-actuator joint forces respectively. Matrix $\bar{\boldsymbol{M}}$ and vector $\boldsymbol{f}^{(na)}$ can be expressed in explicit forms.

For a closed-loop system, Eq. (1.16) can be written as

$$\bar{\boldsymbol{M}}'\ddot{\boldsymbol{\theta}}_{(i)}=\boldsymbol{E}^T(\boldsymbol{\tau}+\boldsymbol{f}^{(na)}-\bar{\boldsymbol{M}}\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)}).$$

If vector $\boldsymbol{\tau}$ is divided into two sets, $\boldsymbol{\tau}_{(d)}$ and $\boldsymbol{\tau}_{(i)}$, corresponding to vectors of dependent and independent joint velocities $\dot{\boldsymbol{\theta}}_{(d)}$ and $\dot{\boldsymbol{\theta}}_{(i)}$, then substitution of Eq. (1.23) into the above equation yields

$$\boldsymbol{\tau}_{(i)}=\bar{\boldsymbol{M}}'\ddot{\boldsymbol{\theta}}_{(i)}+[\boldsymbol{C}_{(d)}^{-1}\boldsymbol{C}_{(i)}]^T\boldsymbol{\tau}_{(d)} -\boldsymbol{E}^T(\boldsymbol{f}^{(na)}-\bar{\boldsymbol{M}}\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)}).$$

Knowing that in general $\boldsymbol{\tau}_{(d)}=\mathbf{0}$ since vector $\boldsymbol{\tau}_{(d)}$ is redundant, then we get

$$\boldsymbol{\tau}_{(i)}=\bar{\boldsymbol{M}}'\ddot{\boldsymbol{\theta}}_{(i)}-\boldsymbol{E}^T(\boldsymbol{f}^{(na)}-\bar{\boldsymbol{M}}\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)}). \qquad (1.31)$$

Equations (1.30) and (1.31) produce a minimal set of linear algebraic equations for actuator forces. It should be noted that vector $\boldsymbol{\tau}$ for an open-loop system and vector $\boldsymbol{\tau}_{(i)}$ for a closed-loop system simply represent all the necessary actuator forces in the system.

6. Equations of Static Equilibrium

The dynamic analysis of a multibody system generally requires an initial configuration at its static equilibrium state. The equations of static equilibrium can be found from the equations of motion by eliminating all the velocity and acceleration dependent terms. The solution to the resultant equations provides the coordinates of the system at the static equilibrium state. These equations can be linearized and solved iteratively using the Newton-Raphson formula.

When absolute coordinates are used, Eqs. (1.9) and (1.3) yield a large set of equations of static equilibrium as

$$\begin{bmatrix}\boldsymbol{g}+\boldsymbol{D}^T\boldsymbol{\lambda}\\ \boldsymbol{\Phi}\end{bmatrix}=\mathbf{0},$$

where vector $\boldsymbol{g}$ contains only the forces and moments that are independent of velocities. The corresponding linearized Newton-Raphson formula is

$$\begin{bmatrix}\dfrac{\partial(\boldsymbol{g}+\boldsymbol{D}^T\boldsymbol{\lambda})}{\partial\boldsymbol{q}} & \boldsymbol{D}^T\\ \boldsymbol{D} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\Delta\boldsymbol{q}\\ \Delta\boldsymbol{\lambda}\end{bmatrix}=-\begin{bmatrix}\boldsymbol{g}+\boldsymbol{D}^T\boldsymbol{\lambda}\\ \boldsymbol{\Phi}\end{bmatrix}. \qquad (1.32)$$

When joint coordinates are used, a smaller set of equations of static equilibrium can be determined. For an open-loop system, setting all of the velocity and acceleration dependent terms in Eq. (1.12) to be zero produces

$$\boldsymbol{B}^T\boldsymbol{g}=\mathbf{0}.$$

The linearized Newton-Raphson formula for the above equation is

$$\left[\frac{\partial(\boldsymbol{B}^T\boldsymbol{g})}{\partial\boldsymbol{\theta}}\right]\Delta\boldsymbol{\theta}=-\boldsymbol{B}^T\boldsymbol{g}. \qquad (1.33)$$

For a closed-loop system, by allowing all of the velocity and acceleration dependent terms in Eq. (1.13) to be zero, Eqs. (1.13) and (1.6) yield

$$\begin{bmatrix}\boldsymbol{B}^T\boldsymbol{g}+\boldsymbol{C}^T\boldsymbol{\nu}\\ \boldsymbol{\Psi}\end{bmatrix}=\mathbf{0}.$$

The linearized Newton-Raphson formula for this set is written as

$$\begin{bmatrix}\dfrac{\partial(\boldsymbol{B}^T\boldsymbol{g}+\boldsymbol{C}^T\boldsymbol{\nu})}{\partial\boldsymbol{\theta}} & \boldsymbol{C}^T\\ \boldsymbol{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\Delta\boldsymbol{\theta}\\ \Delta\boldsymbol{\nu}\end{bmatrix}=-\begin{bmatrix}\boldsymbol{B}^T\boldsymbol{g}+\boldsymbol{C}^T\boldsymbol{\nu}\\ \boldsymbol{\Psi}\end{bmatrix}. \qquad (1.34)$$

In order to find the starting values of the vector of Lagrange multipliers $\boldsymbol{\nu}$ for the Newton-Raphson iteration, we can solve the linear equation

$$\boldsymbol{C}\boldsymbol{C}^T\boldsymbol{\nu}=-\boldsymbol{C}\boldsymbol{B}^T\boldsymbol{g}.$$

If all of the velocity and acceleration dependent terms in Eq. (1.16) are set to zero, another useful equation is obtained as

$$\boldsymbol{E}^T\boldsymbol{B}^T\boldsymbol{g}=\mathbf{0},$$

where its linearized Newton-Raphson formula is expressed as

$$\left[\frac{\partial(\boldsymbol{E}^T\boldsymbol{B}^T\boldsymbol{g})}{\partial\boldsymbol{\theta}_{(i)}}\right]\Delta\boldsymbol{\theta}_{(i)}=-\boldsymbol{E}^T\boldsymbol{B}^T\boldsymbol{g}. \qquad (1.35)$$

It should be noted that Eqs. (1.33)~(1.35) are in general relatively small sets of algebraic equations, and vector $\boldsymbol{C}^T\boldsymbol{\nu}$ is only related to the set of cut joint in a closed-loop system. The coefficient matrices in Eqs. (1.33)~(1.35) can be evaluated either explicitly or numerically. For many problems, however, it is possible to find explicit expressions for the coefficient matrix $[\partial(\boldsymbol{B}^T\boldsymbol{g})/\partial\boldsymbol{\theta}]$, since vector $\boldsymbol{B}^T\boldsymbol{g}$ is known explicitly. The coefficient matrices $[\partial(\boldsymbol{C}^T\boldsymbol{\nu})/\partial\boldsymbol{\theta}]$ and $[\partial(\boldsymbol{E}^T\boldsymbol{B}^T$

$\boldsymbol{g})/\partial\boldsymbol{\theta}_{(i)}]$ can in general be evaluated numerically by finite difference method.

7. Equations of Design Sensitivity

For the purpose of optimal design of multibody systems, a performance cost function subject to performance constraints can be defined as functions of design parameters, the configurations, and the motion of the system. In the case of dual cost functions a trade-off method can be used (Sandgren, Gim and Ragsdell, 1985). If a vector of design parameters is denoted by $\boldsymbol{b}$, then the cost function and the performance constraints can be expressed as

$$\min_{b\in\Omega B} \Psi_0(\boldsymbol{b})$$

subject to

$$\Psi_j(\boldsymbol{b})\begin{cases}=0, & \text{for } j=1,\cdots, e\\ \leq 0, & \text{for } j=e+1,\cdots, l,\end{cases}$$

where $\boldsymbol{\Omega}_B$ is the admissible range of design parameters. In general, the performance cost and constraint functions can be written as

$$\Psi=\Psi(\boldsymbol{q},\ \boldsymbol{v},\ \dot{\boldsymbol{v}},\ \boldsymbol{\lambda},\ t;\ \boldsymbol{b}),$$

where t is the time. The first-order variation of the above equation is written as

$$\delta\Psi=\boldsymbol{L}^T\delta\boldsymbol{b}$$

where $\boldsymbol{L}$ is defined as a matrix of design sensitivity coefficients which is generally dependent on variational variables $\boldsymbol{q}_b$, $\boldsymbol{v}_b$, $\dot{\boldsymbol{v}}_b$, and $\boldsymbol{\lambda}_b$. It is noted that matrix $\boldsymbol{L}$ is required by most optimization algorithms.

In order to find $\boldsymbol{q}_b$, $\boldsymbol{v}_b$, $\dot{\boldsymbol{v}}_b$, and $\boldsymbol{\lambda}_b$, from Eqs. (1.9) and (1.5) the equations of design sensitivity can be written in terms of the absolute accelerations as (Chang and Nikravesh, 1985)

$$\begin{bmatrix}\boldsymbol{M} & \boldsymbol{D}^T\\ \boldsymbol{D} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{v}}_b\\ -\boldsymbol{\lambda}_b\end{bmatrix}=\begin{bmatrix}\boldsymbol{g}_b-\boldsymbol{M}_b\dot{\boldsymbol{v}}+\boldsymbol{D}_b^T\boldsymbol{\lambda}\\ \boldsymbol{\gamma}_b-\boldsymbol{D}_b\dot{\boldsymbol{v}}\end{bmatrix}, \quad (1.36)$$

where $(\ )_b$ stands for $\dfrac{\delta(\ )}{\delta b}$ as the first-order variation. A vector of Lagrange multipliers $\boldsymbol{\lambda}$ can be found from the linear equation

$$\boldsymbol{D}\boldsymbol{D}^T\boldsymbol{\lambda}=\boldsymbol{D}(\boldsymbol{M}\dot{\boldsymbol{v}}-\boldsymbol{g}).$$

Now the equations of design sensitivity, described in terms of the joint accelerations, can be derived directly from the minimal set of differential equations of motion. For an open-loop system, the first-order variation of Eq. (1.12) yields

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{\theta}}_b=(\boldsymbol{B}^T\boldsymbol{g})_b-(\boldsymbol{B}^T\boldsymbol{M}\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}})_b-\bar{\boldsymbol{M}}_b\ddot{\boldsymbol{\theta}}. \quad (1.37)$$

If the first-order variation of reaction forces is needed, then Eq. (1.29) gives

$$\begin{aligned}\boldsymbol{g}_{bj}^{(c)}=&[\boldsymbol{I},\ \boldsymbol{B}_{s1}'^T]_b\ (\boldsymbol{M}_s\dot{\boldsymbol{v}}_s-\boldsymbol{g}_s)\\ &+[\boldsymbol{I},\ \boldsymbol{B}_{s1}'^T]\ (\boldsymbol{M}_s\dot{\boldsymbol{v}}_s-\boldsymbol{g}_s)_b \end{aligned}\quad (1.38)$$

For a closed-loop system, Eqs. (1.13) and (1.8) give rise to the first-order variation as

$$\begin{bmatrix}\bar{\boldsymbol{M}} & \boldsymbol{C}^T\\ \boldsymbol{C} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{\theta}}_b\\ -\boldsymbol{\nu}_b\end{bmatrix}=\begin{bmatrix}(\boldsymbol{B}^T\boldsymbol{g})_b-(\boldsymbol{B}^T\boldsymbol{M}\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}})_b-\bar{\boldsymbol{M}}_b\ddot{\boldsymbol{\theta}}+\boldsymbol{C}_b^T\boldsymbol{\nu}\\ \boldsymbol{\varkappa}_b-\boldsymbol{C}_b\ddot{\boldsymbol{\theta}}\end{bmatrix} \quad (1.39)$$

The vector or Lagrange multipliers $\boldsymbol{\nu}$ for a cut joint(s) is determined from the solution of

$$\boldsymbol{C}\boldsymbol{C}^T\boldsymbol{\nu}=\boldsymbol{C}(\bar{\boldsymbol{M}}\ddot{\boldsymbol{\theta}}-\boldsymbol{f}).$$

In some design problems, it may not be necessary to determine $\boldsymbol{\nu}_b$, therefore Eq. (1.39) can be simplified by eliminating $\boldsymbol{\nu}_b$. Premultiplication of the first row submatrix of Eq. (1.39) by $\boldsymbol{E}^T$ yields

$$\begin{bmatrix}\boldsymbol{E}^T\bar{\boldsymbol{M}}\\ \boldsymbol{C}\end{bmatrix}\ddot{\boldsymbol{\theta}}_b=\begin{bmatrix}\boldsymbol{E}^T[(\boldsymbol{B}^T\boldsymbol{g})_b-(\boldsymbol{B}^T\boldsymbol{M}\dot{\boldsymbol{B}}\dot{\boldsymbol{\theta}})_b-\bar{\boldsymbol{M}}_b\ddot{\boldsymbol{\theta}}+\boldsymbol{C}_b^T\boldsymbol{\nu}]\\ \boldsymbol{\varkappa}_b-\boldsymbol{C}_b\ddot{\boldsymbol{\theta}}\end{bmatrix} \quad (1.40)$$

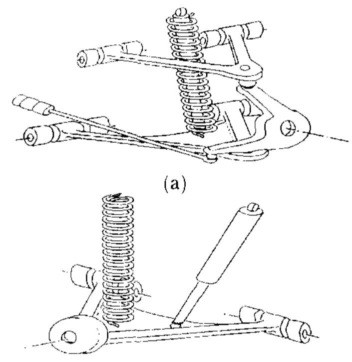
since $\boldsymbol{E}^T\boldsymbol{C}^T=\boldsymbol{0}$ for the constraints of the cut joints. Furthermore, another formula in terms of the generalized accelerations can be derived from Eq. (1.16) as

$$\bar{\boldsymbol{M}}'\ddot{\boldsymbol{\theta}}_{(i)b}=(\boldsymbol{E}^T\boldsymbol{f})_b-(\boldsymbol{E}^T\bar{\boldsymbol{M}}\dot{\boldsymbol{E}}\dot{\boldsymbol{\theta}}_{(i)})_b-\bar{\boldsymbol{M}}'_b\ddot{\boldsymbol{\theta}}_{(i)}. \quad (1.41)$$

The equations of design sensitivity for open- or closed-loop systems, as expressed by Eq. (1.36) in terms of absolute coordinates, provides a large set of differential-algebraic equations. With such a large set of equations, it is impractical to carry out a realistic design sensitivity analysis for any large-scale multibody system. In contrast, when joint coordinates are used, Eq. (1.37) yields a minimal set of differential equations for an open-loop system. For a closed-loop system, Eq. (1.41) yields the minimal set of differential equations, while Eq. (1.39) or (1.40) provides a small set of differential-algebraic equations.

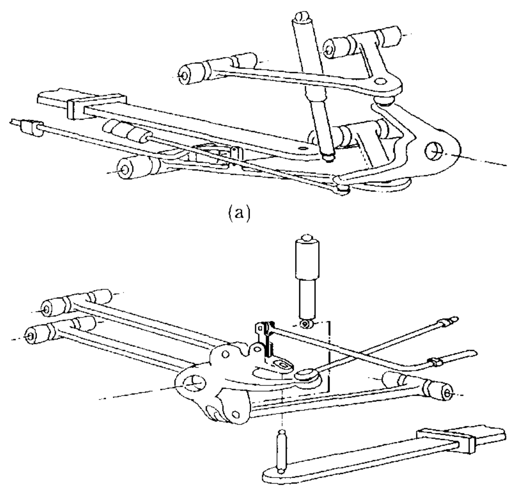
8. Comparison

For the purpose of comparing the problem size using different formulations, two examples are presented here. The first example is a three-dimensional multibody model of a utility truck (Gim, Pereira and Nikravesh, 1987). The model consists of the main chassis, suspension subsystems, steering subsystem, and four wheels. The front wheels are connected to the main chassis by unequal A-arms (double wishbones) as shown in Fig. 1.6(a). The rear wheels are connected to the main chassis by semi-trailing arms as shown in Fig. 1.6(b). Suspension springs, shock absorbers, and jounce stops are modeled by point-to-point spring-damper elements with nonlinear characteristics. The model represents fifteen degrees of freedom; six degrees of freedom correspond to the main chassis, four to the four suspension systems, four to the rolling wheels, and one to the steering. Instead of modeling the steering mechanism as a multibody system, the steering command is enforced on the front wheels using a holonomic constraint. Note that the model consists of thirteen rigid bodies.



(a) (b)

Fig. 1.6 The suspension subsystem of a utility truck (a) front left and (b) rear left



(a) (b)

Fig. 1.7 The suspension subsystem of a sports car (a) front left and (b) rear left

As a second example, a multibody representation of a sports car is described here (Gim, Lankarani and Nikravesh, 1988). The elements that are incorporated in the model consist of the main chassis, suspension subsystems, steering subsystem, and four tires. The system contains several closed kinematic loops and it represents a twenty nine degrees of freedom system. The knuckle of the front left suspension subsystem is attached to the main chassis by a pair of lower and upper control A-arms (LCA and UCA) as shown in Fig. 1.7(a). The A-arms are attached to the chassis by revolute joints and to the knuckle by spherical joints. The revolute joints connecting the UCA to the chassis are housed within elastic bushings. A tie rod is connected between the steering guide rod and the knuckle by spherical joints. Translation of the guide rod causes the steering angle to change. A stabilizer bar connects the two LCA's on the opposite sides through a short link on each side. A lateral leaf-spring, which is fixed to the chassis with two symmetrically placed bushings, has its free ends resting on

Table 1.1 A comparson of number of absolute and joint coordinate formulations of a utility truck

Type of Equations	Absolute Coord. Formulation	Joint Coord. Formulation	
	Number of DAE	Num. of DAE	Num. of ODE
Different. Eqs.	78	22	15
Constraint Eqs.	63	7	0
Eqs. of motion	141	29	15

Note:
DAE: A large set of mixed differential-algebraic equations of motion
ODE: A minimal set of ordinary differential equations of motion

Table 1.2 A comparison of number of absolute and joint coordinate formulations of a conceptual sports car

Type of Equations	Number of DAE in Absolute Coord. Formulation	Number of ODE in Joint Coord. Formulation
Different. Eqs.	66	29
Constraint Eqs.	37	0
Eqs. of motion	103	29

the LCA's on each side. The free ends are not fixed rigidly to the LCA's; they can slide when LCA's move. A shock absorber is also attached between each LCA and the chassis.

The knuckle of the rear left suspension subsystem is attached to the chassis via two trailing links, a lateral strut, and a tie rod as shown in Fig. 1.7(b). The revolute joint on the trailing links and the lateral strut are housed inside elastic bushings. The tie rod is attached to the knuckle and the chassis by spherical joints. A stabilizer bar, which connects the two knuckles on the opposite sides of the vehicle, is attached to each knuckle by a short link. A lateral leaf-spring is fixed to the chassis with two symmetrically placed bushings, and it is attached to each knuckle by a small link. A shock absorber is attached between the chassis and the knuckle. In the model, the axle is not considered since it does not introduce any additional kinematic constraint on the motion of the knuckle. It is assumed that there is a revolute joint between the wheel and the knuclke, and the driving torque is applied to the rear wheels directly for the purpose of simulation.

The elastic bushings in these three subsystems allow small local displacements of each joint which are necessary for the system to move. Elimination of these bushings from the model may cause the suspension system to turn into a structure. Each stabilizer bar is modeled as a compliant member with the force dependent on the relative motion of the two knuckles in the rear or the two LCA's in the front. The steering subsystem in the model is simplified by excluding the tie rods and the steering guide ride, and instead introducing a holonomic constraint between each front wheel and the chassis. Note that the model consists of eleven rigid bodies-- some of the bodies with negligible masses are modeled as massless links.

For each of the above models, the number of equations of motion for transient dynamic analysis, using absolute Cartesian and joint coordinate formulations, is shown in Tables 1.1 and 1.2. It is quite obvious that the size of the problem can be reduced substantially when the joint coordinate formulation is used instead of the absolute Cartesian coordinate formulation. Similar conclusions can be derived when these formulations are compared against each other for the static equilibrium and the design sensitivity analyses.

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